STAT0041: Stochastic Calculus

Lecture 10 - Properties of Itô Integral

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Key concepts:

- *Itô isometry;*
- *martingale property of Itô integral.*

10.1 Basic Properties of Itô Integral

Following proposition is the basic property of Itô Integral. The proof is similar to simple step process case.

Proposition 10.1 *For $H, G \in \mathscr{L}_T^2$ and $a, b \in \mathbb{R}$:*

(1) (Mean zero) $\mathbb{E}\left[\int_0^T H_t \mathrm{d}B_t\right] = 0$;

(2) (Itô Isometry) $\mathbb{E}\left[\left(\int_0^T H_t \mathrm{d}B_t\right)^2\right] = \mathbb{E}\left[\int_0^T |H_t|^2 \, \mathrm{d}t\right]$;

(3) (Linearity) $\int_0^T (aH_t + bG_t)\,\mathrm{d}B_t = a\int_0^T H_t \mathrm{d}B_t + b\int_0^T G_t \mathrm{d}B_t$.

10.2 Itô Integral as a Stochastic Process

Consider $H \in \mathscr{L}_T^2$, for $t \in [0, T]$, we define a stochastic process

$$\mathcal{I}_t[H] := \int_0^t H_s \mathrm{d}B_s := \int_0^T H_s \mathbf{1}_{[0,t]}(s) \mathrm{d}B_s.$$

We call the process $(\mathcal{I}_t[H])_{t\in[0,T]}$ the indefinite integral of H with respect to B.

Following theorem shows that Itô integral is a continuous martingale.

Theorem 10.2 *Let $H \in \mathscr{L}_T^2$. Then the process*

$$\xi_t = \int_0^t H_s dB_s$$

is a $\mathscr{F}_t$ martingale.

Proposition 10.3 *Let $H \in \mathscr{L}^2_T$. Then the process*

$$\eta_t = \left(\int_0^t H_s dB_s \right)^2 - \int_0^t H_s^2 ds$$

is a $\mathscr{F}_t$ martingale.

Remark 10.4 *In this lecture, we proved that Itô integral is a martingale. It is interesting that the converse is also true: Any martingale can be represented as an Itô integral. This result, called the* martingale representation theorem*, is important for many applications.*